Nonmeasurable unions, around Fremlin-Todorcevic theorem

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Theorem (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski)

Let X - Polish space, $\mathcal{I} \subseteq P(X)$ - σ-ideal with Borel base. If

- $\mathcal{J} \subseteq \mathcal{I}$ point-finite family,
  i.e. $\forall x \in X \ \{A \in \mathcal{A} : \ x \in A\}$ is finite,
- $\bigcup \mathcal{J} \notin \mathcal{I}$,

then there exists $\mathcal{J}' \subseteq \mathcal{J}$ such that $\bigcup \mathcal{J}'$ is not $\mathcal{I}$-measurable,
i.e. does not belong to the σ-field generated by σ-ideal $\mathcal{I}$ and σ-field of all Borel subsets $\mathrm{Bor}(X)$.

K. Kuratowski, *A theorem on ideals and some applications of it to the Baire property in Polish spaces*, Uspekhi Matematicheskikh Nauk, 31 (1976), 108–111.

L. Bukovsky, *Any partition into Lebesgue measure zero sets produces a nonmeasurable set*, Bulletin of the Polish Academy of Sciences Mathematics, 27, (1979), 431–435.

J. Brzuchowski, J. Cichoń, E. Grzegorek, Cz. Ryll-Nardzewski, *On the existence of nonmeasurable unions*, Bulletin of the Polish Academy of Sciences Mathematics, 27, (1979), 447–448.

Theorem(Fremlin, Todorcevic)

If $\mathcal{J} \subseteq \mathcal{N}$ is a partition of $[0,1]$ and $\varepsilon > 0$ then there exists $\mathcal{J}' \subseteq \mathcal{J}$ such that

$$\lambda_*(\bigcup \mathcal{J}') < \varepsilon \text{ and } \lambda^*(\bigcup \mathcal{J}') > 1 - \varepsilon.$$

Theorem(Cichoń, Rałowski, Ryll-Nardzewski, Morayne, Żeberski)

If $\mathcal{J} \subseteq \mathcal{M}$ is a partition of $[0,1]$ then there exists $\mathcal{J}' \subseteq \mathcal{J}$ such that $\bigcup \mathcal{J}'$ is completely $\mathcal{M}$-nonmeasurable.

Fremlin, S. Todorcevic, *Partitions of [0,1] into negligible sets*, https://www1.essex.ac.uk/maths/people/fremlin/n04607.ps, (2004).

J. Cichoń, M. Morayne, R. Rałowski, C. Ryll-Nardzewski, S. Żeberski, *On nonmeasurable unions*, Topology and its Applications, 154 (2007), 884-893.

Set-Cover game $\Game_{\mathcal{S},\mathcal{C}}$

Two players S and C on a set X, endowed with two families of subsets $\mathcal{S}$ and $\mathcal{C}$.

The player S starts the game choosing a set $S_0 \in \mathcal{S}$ and the player C answers suggesting a countable cover $\mathcal{C}_0 \subseteq \mathcal{C}$ of S_0.

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At the n-th inning player S selects a set $S_n \in \mathcal{S}$ with $S_n \prec \mathcal{C}_{n-1}$ and player C answers with a countable cover $\mathcal{C}_n \in \mathcal{C}$ of the set S_n.

The player C is declared the winner if $\bigcap_{n\in\omega} S_n \neq \emptyset$.

Main result

Let $\mathcal{I}$ be a σ-ideal on a set X and $\mathcal{S}, \mathcal{C}$ be two families of subsets of a set X such that the player C has a winning strategy in the game $\eth_{\mathcal{S}\setminus\mathcal{I},\mathcal{C}}$. For every point-finite subfamily $\mathcal{J} \subseteq \mathcal{I}$ of cardinality $0 < |\mathcal{J}| \leq \mathfrak{c}$, there exists a Cantor scheme $(\mathcal{J}_s)_{s\in 2^{<\omega}}$ with $\mathcal{J}_\emptyset = \mathcal{J}$ that has the following properties.

1. For any set $S \subseteq X$ with $S \cap \bigcup \mathcal{J} \notin \mathcal{I}$, the set $\{t \in 2^{<\omega} : S \cap \bigcup \mathcal{J}_t \notin \mathcal{I}\}$ is a perfect subtree of the tree $2^{<\omega}$.
2. For any $\sigma \in 2^{<\omega}$ and $S \in \mathcal{S}$ with $S \cap \bigcup \mathcal{J}_\sigma \notin \mathcal{I}$, there exist a sequence $s \in 2^{<\omega}_\sigma$ and a set $C \in \mathcal{C}(S)$ with $C \cap \bigcup \mathcal{J}_s \notin \mathcal{I}$ such that the set $\{t \in 2^{<\omega}_s : \mathcal{S}(C \cap \bigcup \mathcal{J}_t) \not\subseteq \mathcal{I}\}$ is a chain in the tree $2^{<\omega}$.
3. For any $\sigma \in 2^{<\omega}$ and $S \in \mathcal{S}$ with $S \cap \bigcup \mathcal{J}_\sigma \notin \mathcal{I}$, there exist a sequence $s \in 2^{<\omega}$ and a set $C \in \mathcal{C}(S)$ such that $\sigma \subset s$, $C \cap \bigcup \mathcal{J}_s \notin \mathcal{I}$ and $\mathcal{S}(C \cap \bigcup \mathcal{J}_s) \subseteq \mathcal{I}$.

... main result ...

4. For any $\sigma \in 2^{<\omega}$ and $S \in \mathcal{S}(\bigcup \mathcal{J}_\sigma) \setminus \mathcal{I}$ there exist a sequence $s \in 2^{<\omega}_\sigma$ and a set $C \in \mathcal{C}(S \cap \bigcup \mathcal{J}_s) \setminus \mathcal{I}$ such that the set $\{t \in 2^{<\omega}_s : \mathcal{S}(C \cap \bigcup \mathcal{J}_t) \not\subseteq \mathcal{I}\}$ is a chain in the tree $2^{<\omega}$.
5. For any $\sigma \in 2^{<\omega}$ and $S \in \mathcal{S}(\bigcup \mathcal{J}_\sigma) \setminus \mathcal{I}$ there exist a sequence $s \in 2^{<\omega}_\sigma$ and a set $C \in \mathcal{C}(S \cap \bigcup \mathcal{J}_s) \setminus \mathcal{I}$ such that either $\mathcal{S}(C \setminus J) \subseteq \mathcal{I}$ for some $J \in \mathcal{J}$ or $\mathcal{S}(C \cap \bigcup \mathcal{J}_t) \subseteq \mathcal{I}$ for any sequence $t \in 2^{<\omega}$ with $s \subset t$.
6. If the family $\mathcal{S}$ is multiplicative and $\mathcal{I}$-Lindelöf, then there exists a decreasing sequence $(\Sigma_n)_{n\in\omega} \in (\sigma\mathcal{S})^\omega$ such that
   i. for every $n \in \omega$ and $s \in 2^n$ we have $\mathcal{S}(\bigcup \mathcal{J}_s \setminus \Sigma_n) \subseteq \mathcal{I}$;
   ii. for every $S \in \mathcal{S} \setminus \mathcal{I}$ there exist $C \in \mathcal{C}(S) \setminus \mathcal{I}$ such that either $\mathcal{S}(C \cap \Sigma_n) \subseteq \mathcal{I}$ for some $n \in \omega$ or $\mathcal{S}(C \setminus J) \subseteq \mathcal{I}$ for some $J \in \mathcal{J}$.

Definition

Let $\mathcal{I}$ be an ideal on a set X. A family $\mathcal{A}$ of subsets of X is called

- multilplicative iff for any $A, A' \in \mathcal{A}$ $A \cap A' \in \mathcal{A}$;
- $\mathcal{I}$-Lindelöf if for any subset $S \subseteq X$, there exists a countable subfamily $\mathcal{F} \subseteq \mathcal{A}(S)$ such that $\mathcal{A}(S \setminus \bigcup \mathcal{F}) \subseteq \mathcal{I}$;
- $\mathcal{I}$-ccc if each disjoint subfamily $\mathcal{F} \subseteq \mathcal{A} \setminus \mathcal{I}$ is at most countable.

Lemma

Let $\mathcal{I}$ be an ideal on a set X and $\mathcal{A}$ be a family of subsets of X. If $\mathcal{A}$ is $\mathcal{I}$-ccc, then $\mathcal{A}$ is $\mathcal{I}$-Lindelöf.

$S \subseteq X$ is

- $(\mathcal{A}, \mathcal{I})$-measurable if there are sets $A_1, A_2 \in \mathcal{A}$ such that $A_1 \subseteq S \subseteq X \setminus A_2$ and $X \setminus (A_1 \cup A_2) \in \mathcal{I}$;
- $(\mathcal{A}, \mathcal{I})$-saturated if for any $A \in \mathcal{A}$ with $A \cap S \notin \mathcal{I}$ we have $\mathcal{A}(A \cap S) \not\subseteq \mathcal{I}$;
- $(\mathcal{A} \setminus \mathcal{I})$-Ramsey if for every $A \in \mathcal{A} \setminus \mathcal{I}$ there exists $B \in \mathcal{A} \setminus \mathcal{I}$ such that $B \subseteq A \cap S$ or $B \subseteq A \setminus S$.

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Corrolary

Let $\mathcal{I}$ be a σ-ideal on a set X and $\mathcal{A}$ be an $\mathcal{I}$-winning class of subsets of X. Let $\mathcal{J} \subseteq \mathcal{I}$ be a point-finite subfamily such that $|\mathcal{J}| \leq \mathfrak{c}$.

1. If $\bigcup \mathcal{J} \notin \mathcal{I}$, then for some $\mathcal{J}' \subseteq \mathcal{J}$ the union $\mathcal{J}'$ is not $(\mathcal{A}, \mathcal{I})$-saturated and hence is not $(\sigma\mathcal{A}, \mathcal{I})$-measurable.
2. If $\mathcal{A}(\bigcup \mathcal{J}) \not\subseteq \mathcal{I}$, then for some $\mathcal{J}' \subseteq \mathcal{J}$ the union $\bigcup \mathcal{J}'$ is not $(\mathcal{A} \setminus \mathcal{I})$-Ramsey.

Polishable families

X - topological space, $\mathcal{A} \subseteq P(X)$.
$\mathcal{A}$ is Polishable family iff for any $A \in \mathcal{A}$ there are $P_A, f_A, \mathcal{B}_A$ such that

1. P_A is a Polish space, $\mathcal{B}_A$ is a countable topological base of P_A,
2. $f_A : P_A \to X$ is continuous and $f[P_A] = A$,
3. $\{f_A[B] : B \in \mathcal{B}_A\} \subseteq \mathcal{A}$.

Theorem

X - Hausdorff space, $\mathcal{A} \subseteq P(X)$ - multiplicative Polishable family, $\mathcal{I} \subseteq P(X)$ σ-ideal on X. Then C has a winning strategy in $\eth_{\mathcal{A}\setminus\mathcal{I},\mathcal{A}}$.

Topological space X is

- functionally Hausdorff if for any distinct $x, y \in X$ exists map $g : X \to \mathbb{R}$ s.t. $f(x) \neq f(y)$,
- analytic if X is functionally Hausdorff and there exists a continuous surjection $f : P \to X$ for some Polish space P,
- Borel if X is functionally Hausdorff and there exists a continuous bijection $f : P \to X$ for some Polish space P.

Examples of Polishable families

- For every analytic space X, the family $\Sigma_1^1(X)$ of analytic subspaces is multiplicative Polishable family.
- For every Polish space X, the Borel classes $\Sigma_\alpha^0(X), \prod_\alpha^0(X)$ and $\Delta_\beta^0(X)$ $1 \leq \alpha < \beta < \omega_1$ are Polishable and multiplicative.
- For every Borel space X, the family $\Delta_1^1(X)$ of Borel subspaces is multiplicative Polishable family.

MB-representation

Given any family $\mathcal{F}$ of subsets of a set X, consider the families

$$\mathcal{S}_0(\mathcal{F}) = \{A \subseteq X : \forall F \in \mathcal{F}\ \exists H \in \mathcal{F}\ \ (H \subseteq F \setminus A)\}$$

$$\mathcal{S}(\mathcal{F}) = \{A \subseteq X : \forall F \in \mathcal{F}\ \exists H \in \mathcal{F}\ \ (H \subseteq F \cap A\ \vee\ H \subseteq F \setminus A)\}.$$

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Corollary

Let X be a Polish space and a pair $(\mathcal{B}, \mathcal{I})$ be MB-representable by a family $\mathcal{F}$ consisting of Borel sets. Assume that $\mathcal{B}$ contains all Borel sets. Let $\mathcal{J} \subseteq \mathcal{I}$ be a point-finite family of subsets of X such that $\bigcup \mathcal{J} \notin \mathcal{I}$. Then there is a subfamily $\mathcal{J}' \subseteq \mathcal{J}$ such that $\bigcup \mathcal{J}' \notin \mathcal{B}$.

from main theorem

6. If the family $\mathcal{S}$ is multiplicative and $\mathcal{I}$-Lindelöf, then there exists a decreasing sequence $(\Sigma_n)_{n\in\omega} \in (\sigma\mathcal{S})^\omega$ such that
   i. for every $n \in \omega$ and $s \in 2^n$ we have $\mathcal{S}(\bigcup \mathcal{J}_s \setminus \Sigma_n) \subseteq \mathcal{I}$;
   ii. for every $S \in \mathcal{S} \setminus \mathcal{I}$ there exist $C \in \mathcal{C}(S) \setminus \mathcal{I}$ such that either $\mathcal{S}(C \cap \Sigma_n) \subseteq \mathcal{I}$ for some $n \in \omega$ or $\mathcal{S}(C \setminus J) \subseteq \mathcal{I}$ for some $J \in \mathcal{J}$.

Measure case

For any point-finite subfamily $\mathcal{J} \subseteq \mathcal{N}$ and any $\varepsilon > 0$ there exist a Borel set $A \subseteq [0,1]$ of measure $\lambda(A) > 1 - \varepsilon$ and a finite partition $\mathcal{J} = \mathcal{J}_1 \cup \dots \cup \mathcal{J}_n$ such that $\lambda_*(A \cap \bigcup \mathcal{J}_i) = 0$ for $i \in \{1, \dots, n\}$.

Theorem(Fremlin, Todorcevic)

If $\mathcal{J} \subseteq \mathcal{N}$ is a partition of $[0,1]$ and $\varepsilon > 0$ then there exists a partition $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$ such that $\lambda_*(\bigcup \mathcal{J}_i) < \varepsilon$ for $i = 1, 2$.

from main theorem

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Category case

For any point-finite subfamily $J \subseteq \mathcal{M}$ there exists a closed nowhere dense set $D \subseteq [0,1]$ such that for any neighborhood U of D in X there exists a finite partition $\mathcal{J} = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_n$ such that for every $i \in \{1,\dots,n\}$ the set $\bigcup \mathcal{J}_i \setminus U$ contains no non-meager Borel subsets of $[0,1]$.

Thank You for Your Attention!

Taras Banakh, Robert Rałowski, Szymon Żeberski, A set-cover game and nonmeasurable unions, https://arxiv.org/pdf/2011.11342.pdf